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Temperature-Dependent Particle Production and Stimulated Emissions by External Sources

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Exact solutions for transition amplitudes for particle production and stimulated emission by external sources are derived for *finite* temperatures. More precisely, we obtain the expressions for amplitudes for the emission of an arbitrary number of particles by the sources, and corresponding *stimulated* emission processes, when one is dealing with a generalized multiparticle state (rather than the vacuum) at finite temperatures. The solutions are given for spin-0, massive and massless (photons) spin-1, and spin-$\frac{1}{2}$ particles. As applications, we study the process: photon $\rightarrow$ any photons, in the presence of a strong external electromagnetic current, with the net release of a specified energy, and work out the power radiated by a given electromagnetic current distribution, all at finite temperatures. The latter application includes the radiation emitted by a point charged particle at $T \neq 0$ as a special case.

1. INTRODUCTION

We extend our earlier systematic analysis of stimulated emission (Manoukian, 1986) and particle production (Manoukian, 1984) by external sources for *finite* temperatures $T \neq 0$. Such a systematic analysis is certainly lacking in the literature [see Schwinger (1970) and Pardy (1989) for earlier work]. In the finite-temperature case, the so-called vacuum states are replaced by generalized multiparticle states. We give a complete derivation of the corresponding amplitudes, for arbitrary strong sources, *directly* from our earlier results (Manoukian, 1986) given for $T = 0$. The analysis is given for spin-0, massive and massless (photons) spin-1, as well as for spin-$\frac{1}{2}$ particles. As applications, we study the process: photon $\rightarrow$ any photons, in the presence of a strong external electromagnetic current, with the release of a net energy, and also work out the power radiated by a given current

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distribution, all at finite temperatures. The latter application includes the radiation emitted by a point charged particle for $T \neq 0$ as a special case. Finite-temperature effects have been widely applied in recent years to many areas of physics in field theory and particles (Linde, 1979, 1984; Donoghue *et al.*, 1985, and references therein), in gravitation (Linde, 1979; Manoukian, 1990), and in condensed matter physics (Fetter and Walecka, 1971; Manoukian, 1987; Pardy, 1989).

2. PARTICLE PRODUCTION

2.1. Introduction

Consider a real scalar field interacting with an external source $K(x)$. Suppose we have initially N particles, prior to the switching on of the source, and after the source is switched off we end up again with the N particles in the same configurations. We introduce a discretization notation (Schwinger, 1970; Manoukian, 1984) for the momentum variable by setting in the process

$$K_{\mathbf{p}} = (d\omega_{\mathbf{p}})^{1/2} K(p) \tag{1}$$

$$d\omega_p = \frac{d^3\mathbf{p}}{(2\pi)^3 2p^0}, \qquad p^0 = +(\mathbf{p}^2 + m^2)^{1/2} \tag{2}$$

$$K(p) = \int (dx)\, e^{-ipx} K(x) \tag{3}$$

Let $\{\mathbf{p}_1, \mathbf{p}_2, \ldots.\}$ denote the set of all momenta in the convenient discrete-momentum notation. Let $N_1, N_2, \ldots$ denote the number of particles of the N particles having momenta $\mathbf{p}_1, \mathbf{p}_2, \ldots$, where $N_1 + N_2 + \cdots = N$. The transition amplitude in question has been derived in Manoukian (1986), equation (16), and is given by $(T = 0)$

$$\begin{aligned} &\langle N; N_1, N_2, \ldots | N; N_1, N_2, \ldots \rangle^K \\ &= (N_1!\, N_2! \ldots) \sum{}^{*} \frac{(-|K_1|^2)^{N_1 - m_1}}{[(N_1 - m_1)!]^2} \frac{(-|K_2|^2)^{N_2 - m_2}}{[(N_2 - m_2)!]^2} \cdots \frac{\langle 0_+ | 0_- \rangle^K}{m_1!\, m_2! \ldots} \end{aligned} \tag{4}$$

where $\sum^*$ stands for a summation over all nonnegative integers $m_1, m_2, \ldots$ such that $0 \leq m_i \leq N_i$, $i = 1, 2, \ldots$; $K_i \equiv K_{\mathbf{p}_i}$, and $\langle 0_+ | 0_- \rangle^K$ is the familiar vacuum-to-vacuum transition amplitude:

$$\langle 0_+ | 0_- \rangle^K = \exp\left[\frac{i}{2} \int (dx)\,(dx)\, K(x) \Delta_+(x - x') K(x')\right] \tag{5}$$

where

$$\Delta_+(x - x') = i \int d\omega_{\mathbf{p}}\, e^{ip(x - x')} \qquad \text{for} \quad x^0 > x'^0 \tag{6}$$

Nondiagonal transition amplitudes, generalizing the expression in (4), are also derived in Manoukian (1986), equation (16).

Equation (4) may be rewritten in a more compact notation by noting that

$$\sum_{m_1=0}^{N_1} \frac{(-|K_1|^2)^{N_1-m_1}}{(N_1-m_1)!} \frac{N_1!}{(N_1-m_1)!\,m_1!} = \sum_{m_1=0}^{N_1} \frac{(\partial/\partial a_1)^{m_1}}{m_1!} \frac{(a_1)^{N_1}}{(N_1-m_1)!}$$
$$= \frac{(\partial/\partial a_1+1)^{N_1}}{N_1!}(a_1)^{N_1}, \qquad a_1 \equiv [-|K_1|^2] \tag{7}$$

Hence

$$\langle N; N_1, N_2, \ldots | N; N_1, N_2, \ldots \rangle^K = \langle 0_+|0_-\rangle^K \prod_{i=1}^{\infty} \frac{(\partial/\partial a_i+1)^{N_i}}{N_i!}(a_i)^{N_i} \tag{8}$$

$a_i = [-|K_i|^2]$. Temperature dependence is introduced by averaging (Schwinger, 1970) the expression on the left-hand side of (8) with the statistical factor $C\prod_{i=1}^{\infty}(\exp -\beta p_i^0)^{N_i}$, where $p_i^0 = (\mathbf{p}_i^2 + m^2)^{1/2}$, $\beta = 1/kT$, C is a normalization factor, and k is the Boltzmann constant. Using the easily derived equality

$$\sum_{N=0}^{\infty} \sum_{N_1+N_2+\cdots=N} (x_1)^{N_1}(x_2)^{N_2}\cdots = \frac{1}{\prod_{i=1}^{\infty}(1-x_i)} \tag{9}$$

we obtain for C

$$C = \prod_{i=1}^{\infty} (1-e^{-\beta p_i^0}) \tag{10}$$

Therefore, this thermal average of (8) is

$$\langle 0_+|0_-\rangle^K C \sum_{N=0}^{\infty} \sum_{N_1+N_2+\cdots=N} \frac{[e^{-\beta p_i^0}(\partial/\partial a_i+1)]^{N_i}}{N_i!}(a_i)^{N_i} \tag{11}$$

To carry out the summations in (11), we note that

$$e^{-\beta p_1^0 N_1} \frac{(\partial/\partial a_1+1)^{N_1}}{N_1!}(a_1)^{N_1}$$
$$= \int_{-\infty}^{\infty} d\rho_1 \frac{(\rho_1\, e^{-\beta p_1^0})^{N_1}}{N_1!}\left(\frac{\partial}{\partial a_1}+1\right)^{N_1} \delta(\rho_1 - a_1)$$
$$= \int_{-\infty}^{\infty} d\rho_1 \int_{-\infty}^{\infty} \frac{d\gamma_1}{2\pi} \frac{[\rho_1\, e^{-\beta p_1^0}(-i\gamma_1+1)]^{N_1}}{N_1!} e^{i\gamma_1(\rho_1-a_1)} \tag{12}$$

Therefore, for the thermal average in question denoted by $\langle G_+|G_-\rangle^K$ we have

$$\begin{aligned}\langle G_+|G_-\rangle^K &\equiv C\sum_{N=0}^{\infty}\sum_{N_1+N_2+\cdots=N}\left[\prod_{i=1}^{\infty}(e^{-\beta p_i^0})^{N_i}\right]\\ &\quad\times\langle N;N_1,N_2,\ldots|N;N_1,N_2,\ldots\rangle^K\\ &= C\langle 0_+|0_-\rangle^K\sum_{N=0}^{\infty}\prod_{j=1}^{\infty}\left(\int_{-\infty}^{\infty}d\rho_j\int_{-\infty}^{\infty}\frac{d\gamma_j}{2\pi}e^{i\gamma_j(\rho_j-a_j)}\right)\\ &\quad\times\frac{[\sum_{s=1}^{\infty}\rho_s e^{-\beta p_s^0}(-i\gamma_s+1)]^N}{N!}\\ &= C\langle 0_+|0_-\rangle^K\prod_{j=1}^{\infty}\int_{-\infty}^{\infty}d\rho_j\int_{-\infty}^{\infty}\frac{d\gamma_j}{2\pi}\\ &\quad\times\exp[\rho_j e^{-\beta p_j^0}(-i\gamma_j+1)+i\gamma_j(\rho_j-a_j)]\end{aligned}\tag{13}$$

Or

$$\begin{aligned}\langle G_+|G_-\rangle^K &= C\langle 0_+|0_-\rangle^K\prod_{j=1}^{\infty}\int d\rho_j\,\delta(\rho_j-\rho_j e^{-\beta p_j^0}-a_j)\exp\rho_j e^{-\beta p_j^0}\\ &=\langle 0_+|0_-\rangle^K\prod_{j=1}^{\infty}\exp\left[\frac{e^{-\beta p_j^0}}{1-e^{-\beta p_j^0}}a_j\right]\\ &\equiv\langle 0_+|0_-\rangle^K\exp\left[-\int\frac{d\omega_{\mathbf{p}}|K(p)|^2}{e^{\beta p^0}-1}\right]\end{aligned}\tag{14}$$

The expression in (14) may be rewritten in the equivalent form

$$\langle G_+|G_-\rangle^K=\exp\left[\frac{i}{2}\int(dx)\,(dx')\,K(x)\Delta_+(x-x';T)K(x')\right]\tag{15}$$

$$\Delta_+(x-x';T)=\int\frac{(dp)}{(2\pi)^4}e^{ip(x-x')}\left[\frac{1}{p^2+m^2-i\varepsilon}+\frac{2\pi i\delta(p^2+m^2)}{\exp[\beta(\mathbf{p}^2+m^2)^{1/2}]-1}\right]\tag{16}$$

$\varepsilon\to+0$. The expression in (15) coincides with that of Schwinger (1970) obtained by different methods.

Now we are ready to study production at finite temperatures. To this end, we write $K(x)=K_1(x)+K_2(x)$, where the source K_2 is switched on after the source K_1 is switched off. Then

$$\langle G_+|G_-\rangle^K=\langle G_+|G_-\rangle^{K_2}\exp\left[\sum_{\mathbf{p}}(i\tilde{K}_{2\mathbf{p}}^*)(i\tilde{K}_{1\mathbf{p}})\right]\langle G_+|G_-\rangle^{K_1}\tag{17}$$

where

$$\tilde{K}_{\mathbf{p}} = (d\omega_{\mathbf{p}})^{1/2} K(p) \left[\coth \frac{\beta p^0}{2} \right]^{1/2} \tag{18}$$

Upon using the unitarity expansion

$$\langle G_+ | G_- \rangle^K = \sum_{N=0}^{\infty} \sum_{N_1+N_2+\cdots=N} \langle G_+ | N; N_1, N_2, \ldots \rangle^{K_2} \langle N; N_1, N_2, \ldots | G_- \rangle^{K_1} \tag{19}$$

we may infer from (17) that

$$\langle N; N_1, N_2, \ldots | G_- \rangle^K = \langle G_+ | G_- \rangle^K \frac{(i\tilde{K}_{\mathbf{p}_1})^{N_1}}{(N_1!)^{1/2}} \frac{(i\tilde{K}_{\mathbf{p}_2})^{N_2}}{(N_2!)^{1/2}} \cdots \tag{20}$$

$$\langle G_+ | N; N_1, N_2, \ldots \rangle^K = \langle G_+ | G_- \rangle^K \frac{(i\tilde{K}^*_{\mathbf{p}_1})^{N_1}}{(N_1!)^{1/2}} \frac{(i\tilde{K}^*_{\mathbf{p}_2})^{N_2}}{(N_2!)^{1/2}} \cdots \tag{21}$$

where N_i denotes the number of particles with momenta $\mathbf{p}_i$. Hence, from our earlier work (Manoukian, 1984) we may infer that the probability that a source K emits N particles, N_{Δ_1} of which have momenta in $\Delta_1 \subset R^3, \ldots, N_{\Delta_s}$ of which have momenta in $\Delta_s \subset R^3$, where $N_{\Delta_1} + N_{\Delta_2} + \cdots + N_{\Delta_s} = N$, is

$$\frac{(\int_{\Delta_1} d\omega_{\mathbf{Q}} |K(Q)|^2 \coth(\beta Q^0/2))^{N_{\Delta_1}}}{N_{\Delta_1}!} \times \cdots \frac{(\int_{\Delta_s} d\omega_{\mathbf{Q}} |K(Q)|^2 \coth(\beta Q^0/2))^{N_{\Delta_s}}}{N_{\Delta_s}!} \times \exp\left[- \int d\omega_{\mathbf{Q}} |K(Q)|^2 \coth\left(\frac{\beta Q^0}{2}\right)\right] \tag{22}$$

In particular, for $N_{\Delta_1} = N$, $N_{\Delta_2} = \cdots = N_{\Delta_s} = 0$, $\Delta_1 \equiv R^3$, we obtain for the latter a Poisson distribution:

$$\frac{(\int d\omega_{\mathbf{Q}} |K(\mathbf{Q})|^2 \coth(\beta Q^0/2))^N}{N!} \exp\left[- \int d\omega_{\mathbf{Q}} |K(Q)|^2 \coth \frac{\beta Q^0}{2} \right] \tag{23}$$

with the *average* number of particles emitted by the source, at temperature T, given by

$$\langle N \rangle = \int d\omega_{\mathbf{Q}} [K(Q]^2 \coth\left(\frac{\beta Q^0}{2}\right) \tag{24}$$

2.2. Massive Spin-1 Particles

Consider a massive vector field interacting with an external source J^μ. From equation (26) of Manoukian (1986), and equation (14) here, we obtain

$$\langle G_+ | G_- \rangle^J = \langle 0_+ | 0_- \rangle^J \exp\left[- \int \sum_{\lambda=1,2,3} d\omega_{\mathbf{p}} \, |J_\lambda(p)|^2 \frac{1}{e^{\beta p^0} - 1} \right] \tag{25}$$

with

$$\langle 0_+ | 0_- \rangle^J = \exp\left[\frac{i}{2} \int (dx)\,(dx')\, J^\mu(x) \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m^2} \right) \Delta_+(x-x') J^\nu(x') \right] \tag{26}$$

where

$$J_\lambda(p) = e_\mu(p, \lambda)^* J^\mu(p) \tag{27}$$

and for the polarization vectors

$$\sum_{\lambda=1,2,3} e_\mu(p,\lambda) e_\nu(p,\lambda)^* = \left(g_{\mu\nu} + \frac{p_\mu p_\nu}{m^2} \right) \tag{28}$$

$$p^\mu e_\mu(p, \lambda) = 0, \qquad \lambda = 1, 2, 3 \tag{29}$$

$$e^\mu(p,\lambda)^* e_\mu(p, \lambda') = \delta_{\lambda\lambda'}, \qquad \lambda, \lambda' = 1, 2, 3 \tag{30}$$

Equation (25) may be rewritten in the equivalent form

$$\langle G_+ | G_- \rangle^J = \exp\left[\frac{i}{2} \int (dx)\,(dx')\, J^\mu(x) \left(g_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m^2} \right) \Delta_+(x-x'; T) J^\nu(x') \right] \tag{31}$$

where $\Delta_+(x-x'; T)$ is defined in (16).

2.3. Photons

For photons interacting with a conserved external electromagnetic current J^μ: $\partial_\mu J^\mu(x) = 0$, $p_\mu J^\mu(p) = 0$, we then have

$$\langle G_+ | G_- \rangle^J = \exp\left[\frac{i}{2} \int (dx)\,(dx')\, J^\mu(x) D_+(x-x'; T) J_\mu(x') \right] \tag{32}$$

where

$$D_+(x-x'; T) = \int \frac{(dp)}{(2\pi)^4} e^{ip(x-x')} \left[\frac{1}{p^2 - i\varepsilon} + 2\pi i \frac{\delta(p^2)}{\exp(\beta|\mathbf{p}|) - 1} \right] \tag{33}$$

In particular, the probability that N_{Δ_1} photons are emitted with momenta in $\Delta_1 \subset R^3$ and polarization $\lambda_1, \ldots,$ and N_{Δ_k} photons with momenta in $\Delta_k \subset R^3$ and polarization λ_k is

$$\frac{[\prod_{i=1}^{k} (\int_{\Delta_i} d\omega_{\mathbf{Q}_i} |e^\mu(Q_i, \lambda_i)^* J_\mu(Q_i)|^2 \coth(\beta Q_i^0/2))^{N_{\Delta_i}}}{N_{\Delta_i}!}$$
$$\times \exp\left[-\int d\omega_{\mathbf{Q}} |e^\mu(Q, \lambda)^* J_\mu(Q)|^2 \coth\left(\frac{\beta Q^0}{2}\right)\right] \tag{34}$$

where $\lambda_1, \ldots, \lambda_k$ are either 1 or 2, and for the polarizations (Schwinger, 1970) one has

$$\sum_{\lambda=1,2} e^\mu(p, \lambda)\, e^\nu(p, \lambda)^* = g^{\mu\nu} - \frac{p^\mu \bar{p}^\nu + p^\nu \bar{p}^\mu}{p\bar{p}}$$
$$p = (p^0, \mathbf{p}), \qquad \bar{p} = (p^0, -\mathbf{p})$$

2.4. Spin-$\frac{1}{2}$ Particles

To treat the situation with spin-$\frac{1}{2}$ particles, we refer to Section 2.4 of Manoukian (1986), equation (54). To this end, the diagonal transition amplitude for $T=0$ is [see equation (54) in Manoukian (1986)]

$$\begin{aligned} &\langle N; N_{r_1}, N_{r_2}, \ldots | M; N_{r_1}, N_{r_2}, \ldots \rangle^\eta \\ &= \sum{}^* (-1)^{m_{r_1}(2N)} (-1)^{m_{r_2}(2N+2N_{r_1})} \\ &\quad \times (-1)^{m_{r_3}(2N+2N_{r_1}+2N_{r_2})} \cdots (i\eta_{r_1})^{N_{r_1}-m_{r_1}} \cdots \langle 0_+ | 0_- \rangle^\eta \cdots (i\eta_{r_1}^*)^{N_{r_1}-m_{r_1}} \end{aligned} \tag{35}$$

with $r = (p, \sigma, \varepsilon)$ standing for momentum, spin $\sigma = 1,2$, particle/antiparticle $\varepsilon = \mp$, respectively. $\sum^*$ stands for a summation over nonnegative integers m_i, with $0 \le m_i \le N_i$, $N_i = 0$ or 1. Clearly, the phase factors disappear and the summations over the m_i give

$$\begin{aligned} &\langle N; N_{r_1}, N_{r_2}, \ldots | N; N_{r_1}, N_2, \ldots \rangle^\eta \\ &= (1 + \eta_{r_1}^* \eta_{r_1})^{N_1} (1 + \eta_{r_2}^* \eta_{r_2})^{N_2} \cdots \langle 0_+ | 0_- \rangle^\eta \end{aligned} \tag{36}$$

We have to average (36) over the N_i, and N with the statistical factor $C \prod_{i=1}^{\infty} (\exp -\beta p_i^0)^{N_i}$. The easily derived equality

$$\sum_{N=0}^{\infty} \sum_{\substack{N_1+N_2+\cdots=N \\ (N_i=0,1)}} (x_1)^{N_1} (x_2)^{N_2} \cdots = \prod_{i=1}^{\infty} (1 + x_i) \tag{37}$$

gives

$$C = 1 \Big/ \prod_{i=1}^{\infty} (1 + e^{-\beta p_i^0}) \tag{38}$$

From (36) and the equality (37) we then have

$$\begin{aligned}\langle G_+ | G_- \rangle^\eta &= C\langle 0_+ | 0_- \rangle^\eta \prod_{i=1}^{\infty} [1 + e^{-\beta p_i^0}(1 + \eta_{r_i}^* \eta_{r_i})] \\ &= \langle 0_+ | 0_- \rangle^\eta \prod_{i=1}^{\infty} \exp\left[\frac{1}{e^{\beta p_i^0}+1} \eta_{r_i}^* \eta_{r_i}\right] \\ &= \langle 0_+ | 0_- \rangle^\eta \exp \sum_{i=1}^{\infty} \left[\frac{1}{e^{\beta p_i^0}+1} \eta_{r_i}^* \eta_{r_i}\right] \end{aligned} \tag{39}$$

where we have used the fact that $(\eta_{r_i}^* \eta_{r_i})^2 = 0$. In detail, (39) may be rewritten as

$$\langle G_+ | G_- \rangle^\eta = \langle 0_+ | 0_- \rangle^\eta \exp\left[\sum_{\mathbf{p},\sigma,\varepsilon} \frac{1}{e^{\beta p^0}+1} \eta_{\mathbf{p}\sigma\varepsilon}^* \eta_{\mathbf{p}\sigma\varepsilon}\right] \tag{40}$$

Finally, we use the definitions [equations (44)-(47) in Manoukian (1986)]

$$\begin{aligned}\eta_{\mathbf{p}\sigma-}^* &= (2m\, d\omega_{\mathbf{p}})^{1/2} \bar{\eta}(p) u(p, \sigma) \\ \eta_{\mathbf{p}\sigma-} &= (2m\, d\omega_{\mathbf{p}})^{1/2} \bar{u}(p, \sigma) \eta(p) \\ \eta_{\mathbf{p}\sigma+}^* &= (2m\, d\omega_{\mathbf{p}})^{1/2} \bar{v}(p, \sigma) \eta(-p) \\ \eta_{\mathbf{p}\sigma+} &= (2m\, d\omega_{\mathbf{p}})^{1/2} \bar{\eta}(-p) v(p, \sigma) \end{aligned} \tag{41}$$

to rewrite (40) as

$$\langle G_+ | G_- \rangle^\eta = \exp\left[i \int (dx)\,(dx')\, \bar{\eta}(x) S_+(x - x'; T) \eta(x')\right] \tag{42}$$

where

$$\begin{aligned} S_+(x - x'; T) = \int \frac{(dp)}{(2\pi)^4} e^{ip(x-x')}(-\gamma p + m) \\ \times \left[\frac{1}{p^2 + m^2 - i\varepsilon} - \frac{2\pi i \delta(p^2 + m^2)}{\exp[\beta(\mathbf{p}^2 + m^2)^{1/2}]}\right] \end{aligned} \tag{43}$$

In particular, we have from (42) and (41) for the average number of electrons emitted by the source the expression

$$\langle N_{e^-}(T) \rangle = \frac{1}{2} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \left(\frac{2m}{2p^0}\right) \tanh\left(\frac{\beta p^0}{2}\right) \sum_{\sigma} \bar{\eta}(p) u(p, \sigma) \bar{u}(p, \sigma) \eta(p) \tag{44}$$

$$p^0 = +(\mathbf{p}^2 + m^2)^{1/2}$$

3. STIMULATED EMISSIONS

The amplitudes for stimulated emissions at finite temperatures may be found from the work of Section 2 of the present paper and our earlier results in Section 2 of Manoukian (1986). In particular, for spin-0 particles we have (diagonal *and* nondiagonal elements)

$$\begin{aligned}\langle N; N_1, N_2, \ldots | M; M_1, M_2, \ldots\rangle_T^K &\\ = (N_1!\, N_2! \ldots M_1!\, M_2! \ldots)^{1/2} &\\ \times \sum{}^* \frac{(i\tilde{K}_1)^{N_1-m_1}}{(N_1-m_1)!} \frac{(i\tilde{K}_2)^{N_2-m_2}}{(N_2-m_2)!} \cdots \frac{\langle G_+ | G_-\rangle^K}{m_1!\, m_2! \ldots} &\\ \times \cdots \frac{(i\tilde{K}_2^*)^{M_2-m_2}}{(M_2-m_2)!} \frac{(i\tilde{K}_1^*)^{M_1-m_1}}{(M_1-m_1)!} &\end{aligned} \tag{45}$$

where $\sum^*$ stands for a summation over all nonnegative integers m_i such that $0 \leq m_i \leq \min[N_i, M_i]$, $i = 1, 2, \ldots$ and

$$\tilde{K}_i = (d\omega_{\mathbf{p}_i})^{1/2} K(p_i) \left[\coth\left(\frac{\beta p_i^0}{2}\right)\right]^{1/2}, \qquad p_i' = (\mathbf{p}_i^2 + m^2)^{1/2} \tag{46}$$

For massive and massless spin-1 particles, the expression in (45) still holds true with the $\tilde{K}_i$ replaced by

$$\tilde{J}_i = (d\omega_{\mathbf{p}_i})^{1/2}\, e_\mu(p_i, \lambda_i)^* J^\mu(p_i) \left(\coth \frac{\beta p_i^0}{2}\right)^{1/2} \tag{47}$$

with $p_i^0 = (\mathbf{p}_i^2 + m^2)^{1/2}$, $\lambda_i = 1, 2, 3$, $p_i^0 = |\mathbf{p}_i|$, $\lambda_i = 1, 2$, respectively.

For spin 1/2 we have the more complicated expression [see in particular equation (54) in Manoukian (1986) and equation (44) in the present work]

$$\begin{aligned}\langle N; N_{r_1}, N_{r_2}, \ldots | M; M_{r_1}, M_{r_2}, \ldots\rangle_T^\eta &\\ = \sum{}^* (-1)^{m_{r_1}(N+M)} &\\ \times (-1)^{m_{r_2}(N+M+N_{r_1}+M_{r_1})} (-1)^{m_{r_3}(N+M+N_{r_1}+N_{r_2}+M_{r_1}+M_{r_2})} &\\ \times \cdots (i\eta_{r_1})^{N_{r_1}-m_{r_1}} (i\eta_{r_2})^{N_{r_2}-m_{r_2}} \cdots \langle G_+ | G_-\rangle^\eta \cdots &\\ \times \cdots (i\eta_{r_2}^*)^{M_{r_2}-m_{r_2}} (i\eta_{r_1}^*)^{M_{r_1}-m_{r_1}} &\end{aligned} \tag{48}$$

where $\sum^*$ stands for a summation over the m_{r_i} with $0 \leq m_{r_i} \leq \min(N_{r_i}, M_{r_i})$, N_{r_i}, M_{r_i} equal to 0 or 1, and finally

$$\tilde{\eta}_{r_i} = \eta_{r_i} \left\{\tanh\left[\frac{\beta}{2} (\mathbf{p}_i^2 + m^2)^{1/2}\right]\right\}^{1/2} \tag{49}$$

4. APPLICATIONS

As an application we consider the process: photon $\to$ any photons, in the presence of a strong external electromagnetic current, with the net release of a specified energy ω at finite temperature. To study this radiation process, we use the result in (45) with (46). Clearly, the connected process where the initial photon gets first absorbed by the current source is

$$\langle N; N_1, N_2, \ldots | 1_1 \rangle^J_{c,T} = \frac{(i\tilde{J}_1)^{N_1}}{(N_1!)^{1/2}} \frac{(i\tilde{J}_2)^{N_2}}{(N_2!)^{1/2}} \cdots \langle G_+ | G_- \rangle^J (i\tilde{J}_1^*) \tag{50}$$

Suppose the net energy release is ω; then the probability of the radiation process is

$$P(\omega) = \sum_{N=0}^{\infty} \sum_{N_1+N_2+\cdots=N} \frac{(|\tilde{J}_1|^2)^{N_1}}{N_1!} \frac{(|\tilde{J}_2|^2)^{N_2}}{N_2!} \cdots |\langle G_+ | G_- \rangle^J|^2 |\tilde{J}_1|^2$$
$$\times \delta(N_1|\mathbf{p}_1| + N_2|\mathbf{p}_2| + \cdots - |\mathbf{p}_1| - \omega) \tag{51}$$

Upon using the integral expression

$$\delta(N_1|\mathbf{p}_1| + N_2|\mathbf{p}_2| + \cdots - |\mathbf{p}_1| - \omega)$$
$$= \int_{-\infty}^{\infty} \frac{dx}{2\pi} \exp\{i[(N_1 - 1)|\mathbf{p}_1| + N_2|\mathbf{p}_2| + \cdots - \omega]x\} \tag{52}$$

the expression in (51) is explicitly summed to

$$P(\omega) = |\tilde{J}_1|^2 \int_{-\infty}^{\infty} \frac{dx}{2\pi} \exp[-i(\omega + |\mathbf{p}_1|)x]$$
$$\times \exp\left[\sum_i |\tilde{J}_i|^2 \exp(i|\mathbf{p}_i|x)\right] |\langle G_+ | G_- \rangle^J|^2 \tag{53}$$

and hence from (32) we finally have

$$P(\omega) = |e^*_\mu(p_1, \lambda_1) J^\mu(p_1)|^2 \frac{d^3\mathbf{p}_1}{(2\pi)^3 2|\mathbf{p}_1|} \int_{-\infty}^{\infty} \frac{dx}{2\pi} \exp[-i(\omega + \mathbf{p}_1)x]$$
$$\times \exp\left\{-\int \frac{d^3p}{(2\pi)^3 2|\mathbf{p}|} [J^\mu(p)^* J_\mu(p)]\right.$$
$$\left.\times \coth\left(\frac{\beta}{2}|\mathbf{p}|\right)[1 - \exp(i|\mathbf{p}|x)]\right\} \tag{54}$$

where, with the exception of the total radiated energy ω, the momenta and polarizations of the final products are not measured.

As a second application, we consider the power radiated by a given external current distribution. To this end, the average number of photons emitted by the external current is, from (34) and (24),

$$\langle N\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \coth\left(\frac{\beta|\mathbf{k}|}{2}\right) J^\mu(k)^* J_\mu(k) \tag{55}$$

Using for $k^0 = |\mathbf{k}|$ the relation

$$J^0(k) = \frac{k^i}{|\mathbf{k}|} J^i(k) \tag{56}$$

we obtain

$$\langle N\rangle = \int_0^\infty N(\omega)\, d\omega \tag{57}$$

with $|\mathbf{k}| = \omega$, where

$$N(\omega) = \frac{\omega}{16\pi^3} \coth\left(\frac{\beta\omega}{2}\right) \int d\Omega\, \hat{J}^i(\omega, \mathbf{n})^* \hat{J}^i(\omega, \mathbf{n}) \tag{58}$$

$$J^i(k) = \int dt \int d^3\mathbf{x}\, \{\exp[i\omega(t - \mathbf{n}\cdot\mathbf{x})]\} J^i(x)$$

$$\equiv J^i(\omega, \mathbf{n}) = J^i(-\omega, \mathbf{n})^* \tag{59}$$

$$\hat{J}^i(\omega, \mathbf{n}) = (\delta^{ij} - n^i n^j) J^j(\omega, \mathbf{n}) \tag{60}$$

Here $N(\omega)$ denotes the average number of photons with energy ω per unit interval. Accordingly, the total energy radiated is

$$\mathscr{E} = \frac{1}{16\pi^3} \int d\Omega \int_0^\infty d\omega\, \omega^2 \coth\left(\frac{\beta\omega}{2}\right) \hat{J}^i(\omega, \mathbf{n})^* \hat{J}^i(\omega, \mathbf{n}) \tag{61}$$

In the spirit of the work of Schwinger *et al.* (1976), the total radiated power is, upon symmetrization over ω,

$$P(t) = \frac{1}{16\pi^2} \int d\Omega \left| \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-i\omega t} \omega \left(\left| \coth \frac{\beta\omega}{2} \right|^{1/2} \right) \hat{J}^i(\omega, \mathbf{n}) \right|^2 \tag{62}$$

for finite temperatures.

For low and high temperatures we may use, respectively, the expansions

$$\coth x = \begin{cases} 1 + 2e^{-2x} + \cdots, & x \to \infty \\ -(1 + 2e^{2x} + \cdots), & x \to -\infty \end{cases} \tag{63}$$

$$\coth x = \frac{1}{x} + \frac{x}{3} + \cdots, \qquad x \to 0 \tag{64}$$

giving

$$P_0(t) = \frac{1}{16\pi^2} \int d\Omega \left| \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \omega \hat{J}^i(\omega, \mathbf{n}) \right|^2, \qquad T \to 0 \tag{65}$$

$$P_\infty(t) = \frac{kT}{4\pi^2} \int d\Omega \left| \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} (\operatorname{sgn} \omega) |\omega|^{1/2} \hat{J}^i(\omega, \mathbf{n}) \right|^2, \qquad T \to \infty \tag{66}$$

Of particular interest is the slow motion of a point charged particle with instantaneous velocity $v^i(t)$, which amounts to writing

$$\int d^3\mathbf{x}\, [\exp(-i\mathbf{k} \cdot \mathbf{x})] J^i(x) = \int d^3\mathbf{x}\, J^i(x) = ev^i(t) \tag{67}$$

giving, upon performing the angular integration, *at* temperature T, the expression

$$P(t) = \frac{2}{3}\left(\frac{e^2}{4\pi}\right) \left| \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \omega \left| \coth\left(\frac{\beta\omega}{2}\right) \right|^{1/2} \tilde{v}^i(\omega) \right|^2 \tag{68}$$

where

$$\tilde{v}^i(\omega) = \int_{-\infty}^{\infty} dt'\, e^{i\omega t'} v^i(t) \tag{69}$$

In particular we have for (68)

$$P_0(t) = \frac{2}{3}\left(\frac{e^2}{4\pi}\right) |\dot{v}^i(t)|^2, \qquad T \to 0 \tag{70}$$

which is the classic Larmor formula (Panofsky and Phillips, 1962; Schwinger, 1949), and

$$P_\infty(t) = \frac{8}{3}\left(\frac{e^2}{4\pi}\right) kT \left| \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (\operatorname{sgn} \omega) |\omega|^{1/2} \tilde{v}^i(\omega) \right|^2, \qquad T \to \infty \tag{71}$$

Other applications are similarly carried out.

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